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# Radical 2-subgroups of the Monster and the Baby Monster

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**Abstract**

The radical 2-subgroups of the Monster and the Baby Monster are determined up to conjugacy, which completes the classification problem of radical 2-subgroups for all sporadic simple groups.
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## Introduction

Let $p$ be a prime dividing the order of a finite group $G$. A nontrivial $p$-subgroup $R$ of $G$ is called *radical* if $R = O_p(N_G(R))$, where $O_p(X)$ denotes the largest normal $p$-subgroup of a group $X$. In this paper, the radical 2-subgroups of the Monster and the Baby Monster are determined up to conjugacy, based on the remarkable classification by Meierfrankenfeld and Shpectorov of maximal 2-local subgroups of these groups [14,15]. Together with previous works, this completes the classification of radical 2-subgroups of all sporadic simple groups. To emphasize the completeness, literature for each sporadic group is given in the appendix, though the list may not be comprehensive. There are included the lists of radical 2-subgroups for $J_2$ and $McL$, as we are unaware of any published works for them.

See the introductions of [22] and [13] for the motivations and the strategy to determine the radical $p$-subgroups. Note that classification of the radical subgroups is required to

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verify Dade's conjecture. For the Monster and the Baby Monster, their maximal 2-local subgroups are neatly described in [15] and [14], which allows us to carry out the classification of their radical 2-subgroups rather smoothly. We need some fusion arguments to examine the candidates for radical 2-subgroups whose normalizers are contained in a maximal 2-local subgroup in the Baby Monster which is not of "characteristic-2" type. Except this point (see Section 1.2), all the information we need can be found in [15] and [14]. In that sense, this paper is just a corollary of their remarkable classification.

The main results are summarized in Tables 1 and 2. There, for a representative $R$ of each class of the radical 2-subgroups, we give a brief description of structure of $R$ as well as those of the center $Z(R)$ and the automizer $N_G(R)/R$. Following [15], we use the symbol $\sim$ (e.g., $L_3 \sim 2_+^{1+24}Co_1$) as shorthand for "has structure," while $\cong$ as usual stands for "is isomorphic." To describe the structure of a group, we follow the Atlas notation, in particular, $A.B$ (or $AB$) means an extension of $B$ by $A$ (a group with a normal subgroup isomorphic to $A$ and the factor group isomorphic to $B$). Classes of radical subgroups with isomorphic centers are arranged in the tables according to the fusion of the elements in the center, with class names following the Atlas notation. If fusion is not indicated, the center is identical with the group in the previous row.

Recall that a radical $p$-subgroup $R$ is called *centric*, if any $p$-element centralizing $R$ lies in $Z(R)$. For the importance of centric radical subgroups, see [19] and [20, §4]. We also refer to which representatives are centric, without giving any verification, as it is easy to see.

The symbols $S_n$ and $A_n$ are used to denote the symmetric and alternating groups of degree $n$. In Table 2, the direct product $S_3 \times S_3$ and $S_3 \times S_3 \times S_3$ are abbreviated to $S_3^2$ and $S_3^3$. We also use the symbols $n$, $p^n$ and $2_\varepsilon^{1+2n}$ to denote, respectively, the cyclic group of order $n$, the elementary abelian group of order $p^n$ and the extraspecial group of order $2^{1+2n}$ of $\varepsilon$ type ($\varepsilon = \pm 1$). Furthermore, the symbols $D_8$, $Q_8 \cong 2_-^{1+2}$ and $SD_{16}$ indicate the dihedral, quaternion and semidihedral group of order 8, 8 and 16, respectively.

For a conjugacy class $pX$ of an element of order $p$, an elementary abelian $p$-subgroup is called *$pX$-pure* if all its nontrivial elements lie in $pX$. I also call a subgroup $H$ of $G$ a *$X$-subgroup*, if $H \cong X$.

## 1. Radical 2-subgroups of the Baby Monster

### 1.1. 2-local subgroups of BM

There are four conjugacy classes of involutions in $B := BM$, the Baby Monster, called $2A$, $2B$, $2C$ and $2D$, with centralizers of shapes $2 \cdot {}^2E_6(2).2$, $2_+^{1+22} \cdot Co_2$, $(2^2 \times F_4(2)).2$, and $2^9.2^{16}\Omega_8^+(2).2$, respectively. It follows from [15, Theorem 2] and [14, Theorem B] that every 2-local subgroup of $B$ is contained in one of the following eight subgroups up to conjugacy:

$L_1 \sim 2 \cdot {}^2E_6(2).2$;
$L_2 \sim (2^2 \times F_4(2)).2$;

$L_3 \sim S_4 \times {}^2F_4(2)$;
$L_4 \sim 2_+^{1+22} \cdot Co_2$;
$L_5 \sim 2^2.2^{10}.2^{20}.(M_{22}{:}2 \times S_3)$;
$L_6 \sim 2^3.[2^{32}].(S_5 \times L_3(2))$;
$L_7 \sim 2^5.[2^{25}].L_5(2)$; and
$L_8 \sim 2^9.2^{16}Sp_8(2)$.

We set $V_i := O_2(L_i)$ ($i = 1, \ldots, 8$). By maximality of $L_i$, each $V_i$ is a radical 2-subgroup of $B$ and $L_i = N_B(Z(V_i))$.

We identify the Baby Monster $B$ with a section of the Monster $M$: namely, $B = C_M(t)/\langle t \rangle$ for a $2A$-involution $t$ in $M$. We fix such an involution $t$ throughout this section. Let $H := C_M(t)$ and $T := \langle t \rangle$. We first review several facts established in [15] and [14]. Following these papers, we sometimes refer to $2A$- and $2B$-involutions in $M$ as the nonsingular and singular involutions, respectively. For a singular involution $z \in M$, we denote $Q_z := O_2(C_M(z)) \cong 2_+^{1+24}$. A $2B$-pure subgroup $U$ of $M$ is called *singular*, if $U \subseteq Q_u$ for every $u \in U^\#$. We set $Q_U := \bigcap_{u \in U^\#} Q_u$ for a singular subgroup $U$ of $M$.

We first describe the preimages in $H$ of four classes of involutions in $B$. Let $z$ be a singular involution of $M$. There are two conjugacy classes of involutions in $Q_z \setminus \langle z \rangle$ under the action of $C_M(z)$, the singular and the nonsingular involutions. Thus there is a singular involution $u$ with $t \in Q_u$, for which two involutions $t$ and $tu$ are conjugate under $Q_u$, as $Q_u$ is extraspecial. The image of these two involutions in $B$ is a $2B$-involution. There is a class of nonsingular involutions in $C_M(z) \setminus Q_z$. Thus there is a singular involution $u$ with $t \in C_M(u) \setminus Q_u$. Then $t$ and $tu$ are conjugate in $M$ [14, Lemma 3.1(a)]. The image of these two involutions in $B$ is a $2D$-involution. There is a $2A$-pure (purely nonsingular) fours subgroup of $H$ containing $t$, whose image in $B$ is a $2A$-involution. Furthermore, there is a cyclic subgroup of order four containing $T$, whose image in $B$ is a subgroup generated by a $2C$-involution.

In $M$, there are six classes of singular subgroups: the representatives are of order $2^i$ for $i = 1, \ldots, 5$, and the singular $2^5$-subgroups of $M$ split into two classes $\mathcal{S}(5,1)$ and $\mathcal{S}(5,2)$ [15, Proposition 4.15]. In [15], the following facts about the singular subgroups $U$ of $H$ with $t \in Q_U$ are established (see Lemma 4.4, Corollaries 4.6, 4.9, and 4.11 and Lemma 4.14 of [15]): there are five classes of singular subgroups $U$ with $t \in Q_U$ under $H$. There is no $2^5$-subgroup $U$ of $\mathcal{S}(5,2)$ with $t \in Q_U$ [15, Lemma 4.14(2)], but the subgroups $U$ in $\mathcal{S}(5,1)$ with $t \in Q_U$ form a (nonempty) single conjugacy class under $H$. For each $i = 1, \ldots, 4$, there is a unique class of singular $2^i$-subgroups $U$ with $t \in Q_U$ under $H$.

Those singular subgroups are represented by subgroups of an *ark* containing $t$, that is, a $2^{10}$-subgroup $V$ with $N_M(V) \cong 2^{10+16}\Omega_{10}^+(2)$ [15, Section 5]. There is a quadratic form $Q$ on $V$ preserved by $N_M(V)/C_M(V) \cong \Omega_{10}^+(2)$. Singular (respectively nonsingular) involutions in $V$ coincide with singular (respectively nonsingular) nonzero vectors of $V$ with respect to $Q$. For $v, z \in V^\#$ with $z$ singular, we have $v \in Q_z$ if and only if $v$ is perpendicular to $z$ with respect to the symplectic bilinear form $b_Q$ associated with $Q$. For a singular vector $z$ of $V$, it is perpendicular to $t$ if and only if $zt$ is nonsingular. Thus the image of $z$ in $B$ is a $2B$-involution. Let $U_i$ ($i = 1, 2, 3, 4$) be a singular $2^i$-subgroup of $V$ perpendicular to $t$. We also choose $U_1 \leqslant U_2 \leqslant U_3 \leqslant U_4$. Then we may take $L_{3+i}$ ($i = 1, 2, 3$) so that the preimage $\tilde{Z}(V_{3+i})$ in $H$ of the center $Z(V_{3+i})$ coincides with $T \times U_i$, which is a subgroup of $V$ ($i = 1, 2, 3$). The two singular $2^5$-subgroups of $V$ containing $U_4$ represent $\mathcal{S}(5, 1)$ and $\mathcal{S}(5, 2)$ (but none of them is perpendicular to $t$). The last argument in proof of Theorem 2 in [15] shows that $V$ is a unique ark containing $t$ and $U_4$. Namely,

**Lemma 1.** *For a singular $2^4$-subgroup $U$ of $H$ with $t \in Q_U$, there exists a unique ark containing $t$ and $U$.*

Let $t^\perp$ be the hyperplane of $V$ perpendicular to $t$ with respect to $b_Q$. We may take $V/T = Z(V_8) \cong 2^9$, which contains a hyperplane $t^\perp/T \cong 2^8$ normalized by $N_B(V/T)$. There is a symplectic form on $t^\perp/T$ inherited from $b_Q$ and preserved by $N_B(V/T)$. With respect to this symplectic form, $(Z(V_4), Z(V_5), Z(V_6), U_4T/T)$ is a maximal flag of totally isotropic subspaces of $t^\perp/T$. Observing $N_M(V)/C_M(V) \cong \Omega^+_{10}(2)$, we find that the normalizer $N_B(V/T)$ induces on $V$ the full stabilizer $O_9(2) \cong Sp_8(2)$ of the nonsingular point $T$. Thus we have $L_8 = N_B(V/T)$ and $C_B(V/T)/(V/T) \cong 2^{16}$. This gives an account for the shape of $L_8$ as well as its maximal parabolic subgroups.

Non-characteristic-2 type maximal locals $L_i$ ($i = 1, 2, 3$) are described as follows [14]. As $V_1$ is generated by a $2A$-involution in $B$, its preimage $\tilde{V}_1$ in $H$ is a $2A$-pure $2^2$-subgroup. There is a unique class of outer involutions of $(L_1/V_1)' \cong {}^2E_6(2)$ with the centralizer $F_4(2)$ in $(L_1/V_1)'$ [5, (19.9)(iii)]. Let $g$ be an element of $L_1$ inducing such an automorphism. Then the preimage $D$ in $H$ of $V_1\langle g\rangle$ is a dihedral group $D_8$ containing $\tilde{V}_1 \cong 2^2$. Then $D/T \cong 2^2$ can be taken as $V_2$. With this choice, $L_2' = L_1 \cap L_2 = C_{L_2}(V_2) \cong 2^2 \times F_4(2)$ is the subgroup of $L_2$ stabilizing two $2^2$-subgroups of $D$. Moreover, the preimage of $L_2^\infty \cong F_4(2)$ in $H$ is the centralizer in $H$ of $D$ [14, Lemma 5.4(e)]. There is a semidihedral group $SD_{16}$ containing $D$, which induces an involutive outer automorphism on $L_2^\infty$. Let $A$ be its unique $Q_8$-subgroup. Then $A/T \cong 2^2$ can be taken as $V_3$, and with this choice, $DA/T \cong D_8$ is a Sylow 2-subgroup of the $S_4$-direct factor of $L_3$. The ${}^2F_4(2)$-factor is the stabilizer in $L_2^\infty \cong F_4(2)$ of an outer involution induced by some element of $A/T$.

By the description above, we may assume that $L_3^\infty \cong {}^2F_4(2)'$ is a subgroup of $L_2^\infty \cong F_4(2)$, and $L_2^\infty$ is a subgroup of $L_1' \cong 2.{}^2E_6(2)$.

### 1.2. Subgroups of non-characteristic-2 type maximal locals

In this subsection, we examine the centers of unipotent radicals of $L_3^\infty \cong {}^2F_4(2)'$ and those of the $F_4(2)$-direct factor of $L_2$. For this purpose, we briefly review the embedding ${}^2F_4(2) \subset F_4(2) \subset {}^2E_6(2)$.

The group ${}^2E_6(2)$ is the centralizer in $E_6(4)$ of the commuting product of the graph and the field automorphisms. It is generated by root subgroups $U_r$ for all roots $r$ of the root system of type $F_4$, where the additive group $U_r$ is parametrized by $GF(2)$ or $GF(4)$, according as $r$ is long or short. The subgroup of ${}^2E_6(2)$ generated by root elements $x_r := U_r(1)$ for all roots $r$ is isomorphic to $F_4(2)$, which is the centralizer in ${}^2E_6(2)$ of the graph (and hence the field) automorphism. Let $U$ be a Sylow 2-subgroup of $F_4(2)$ generated by root elements $x_i = x_{r_i}$ for 24 positive roots $r_i$ ($i = 1, \ldots, 24$). We adopt the numbering in [11, Section 2], where $r_5$, $r_2$, $r_1$ and $r_{10}$ are the fundamental roots corresponding to the nodes of the Dynkin diagram of type $F_4$ from left to right, with $r_5$ long. We identify ${}^2F_4(2)$

with a subgroup $F$ of $F_4(2)$ fixed by the graph automorphism $\gamma$ normalizing $U$. In [11, Section 2], the commutator relations among $x_i$ are given as well as the explicit action of $\gamma$. (Note that the root $ar_5 + br_2 + cr_1 + dr_{10}$ is denoted by $abcd$ in [5], while it is denoted by $cbad$ in [11]. Thus the roots $r$, $s$, $\alpha$ and $\beta$ in [5, (13.1), (14.1)] correspond to $r_{24}$, $r_{21}$, $r_{16}$ and $r_{22}$ respectively in [11, Section 2].) We set $x_{i,j} := x_i x_j$ for $i, j = 1, \dots, 24$.

Now $U$ is a Borel subgroup of $F_4(2)$. We denote by $P_i$ the maximal parabolic subgroup of $F_4(2)$ containing $U$, which corresponds to the $i$th node from left for $i = 1, \dots, 4$. The structures of $P_1$ and $P_4$ as well as the corresponding maximal parabolics in ${}^2E_6(2)$ are described in [7, (4.5), (4.6)]. As calculated in [11, (3.3), (3.2), (3.4)], the centralizers of involutions $x_{24}$, $x_{21}$ and $x_{21,24}$ in $F_4(2)$ respectively coincide with $P_1$, $P_4$ and the parabolic subgroup $P_{14} = P_1 \cap P_4$. As $\gamma$ interchanges $P_1$ and $P_4$, $\gamma$ centralizes $x_{21,24}$. Moreover, $\gamma$ induces an outer involutive automorphism on the Levi part $Sp_4(2) \cong S_6$ of $P_{14}$. In particular, $C_F(x_{21,24})$ contains an element of order 5, which implies that $x_{21,24}$ is a $2A$-involution of $F$. Remark here that $F$ has just two classes of involutions, called $2A$ and $2B$, with centralizers $2[2^8](5.4 \times 2)$ and $[2^8](S_3 \times 2)$, respectively [5, (18.6)].

As $\gamma$ interchanges $P_2$ and $P_3$, it also acts on $P_{23} = P_2 \cap P_3$. It is straightforward to verify that $F \cap P_{14} = C_F(x_{21,24})$ and $F \cap P_{23} = N_F(\langle x_{21,24}, x_{17,23}\rangle)$ are two maximal parabolic subgroups of $F$ containing the Borel subgroup $F \cap U$, which is of order $2^{12}$. One can also verify that $\langle x_{21,24}\rangle$ (respectively $\langle x_{21,24}, x_{17,23}\rangle$) is the center of the unipotent radical of $F \cap P_{14}$ (respectively $F \cap P_{23}$). Remark that $\langle x_{21,24}, x_{17,23}\rangle$ is a $2A$-pure fours subgroup in $F$, as $F \cap P_{23}$ induces $S_3$ on it.

**Lemma 2.** *A $2A$-involution of $L_3^\infty \cong {}^2F_4(2)'$ is fused to a $2B$-involution of $B$.*

**Proof.** Let $\tilde{V}_2$ be the preimage of $V_2$ in $H$. As we saw in the previous subsection, $\tilde{V}_2 = \langle t, u, f\rangle \cong D_8$ for some element $f$ of order 4 squaring to $t$, where $\tilde{V}_1 = \langle t, u\rangle$ is a $2A$-pure fours subgroup whose image in $B$ is $V_1$. Moreover, $C_H(u, f)T/T = L_2^\infty \cong F_4(2)$.

By [5, (13.1)], the classes $2A$, $2B$, $2C$ and $2D$ of involutions of $L_2^\infty \cong F_4(2)$ are represented by $x_{24}$, $x_{21}$, $x_{21,24}$ and $x_{16,22}$, respectively. Furthermore, if we consider $F_4(2)$ as a subgroup of ${}^2E_6(2)$, it follows from the description of the fourth maximal parabolic subgroup of ${}^2E_6(2)$ [7, (4.6)] that the former three representatives are contained in a $2^8$-subgroup, which is the center of the unipotent radical of a maximal parabolic of shape $2^{8+16}\Omega_8^-(2)$.

We will see that this center is given by the image $V/\tilde{V}_1$ of an ark $V$ containing $\tilde{V}_1$. Such an ark $V$ exists by [14, Lemma 3.2(b)]. Then $\tilde{V}_1$ is an anisotropic 2-subspace of $V$ with respect to the quadratic form preserved by $N_M(V)/C_M(V) \cong \Omega_{10}^+(2)$. The vectorwise stabilizer $C_H(t, u) \cap N_M(V)$ in $N_M(V)$ of $\tilde{V}_1$ has the shape $C_M(V).\Omega_8^-(2)$, and thus its image in $C_M(\tilde{V}_1)/\tilde{V}_1 \cong {}^2E_6(2)$ is a maximal parabolic subgroup of shape $2^{8+16}\Omega_8^-(2)$, in which $V/\tilde{V}_1$ is the center of the unipotent radical.

Hence from the conclusion above we may assume that $\langle x_{24}, x_{21}\rangle \leqslant V$ and $\langle t, u\rangle \leqslant V$. Let $z = x_{21,24}$. It follows from [14, Lemma 3.6(b), Lemma 3.5] that a $2^3$-subgroup $\langle t, u, z\rangle$ satisfies the case (a) or (b) in [14, Lemma 3.5]. Furthermore, as $z$ lies in the preimage $C_H(t, u, f)$ of $L_2^\infty$ and $u^f = tu$, two involutions $uz$ and $tuz$ in $\langle t, u\rangle z$ are conjugate in $H$. Thus we conclude that exactly one of $\{z, tz\}$ is singular and the other is nonsingular. Hence the image of $z$ in $B$ is a $2B$-involution. (The same argument shows that $x_{21}$ and $x_{24}$ are $2B$-involutions as well. We may also shows that $x_{16,22}$ is a $2D$-involution.) $\square$

**Lemma 3.**

(1) *Let $P$ be a parabolic subgroup of the ${}^2F_4(2)$-direct factor of $L_3$. Then the center of its unipotent radical is the image in $B$ of a singular $2^i$-subgroup $E$ of $H$ with $t \in Q_E$ for $i = 1$ or 2.*

(2) *Let $P$ be a parabolic subgroup of $L_2^\infty \cong F_4(2)$ normalized by an element $g$ of $L_2$ inducing an outer involutive automorphism on $L_2^\infty$. Then $P\langle g\rangle$ normalizes a singular $2^i$-subgroup $E$ of $H$ with $t \in Q_E$ for $i = 1$ or 2.*

**Proof.** (1) Let $F$ be the ${}^2F_4(2)$-direct factor of $L_3$. With the previous notation, a parabolic subgroup $P$ of $F$ is conjugate to $U \cap F$, $P_{14} \cap F$ or $P_{23} \cap F$. In the former two cases, the center of the unipotent radical of $P$ is a subgroup of order 2 generated by $x_{21,24}$. As this involution is a $2B$-involution in $B$ by Lemma 2, the claim follows in this case. In the last case where $P = P_{23} \cap F$, the center of its unipotent radical is $\bar{Z} := \langle x_{24,21}, x_{23,17}\rangle$, which is a $2A$-pure fours subgroup of $F$. By Lemma 2, the involutions in $\bar{Z}$ lie in the class $2B$ of $B$. Thus the preimage of each involution contains two involutions in $H$ other than $t$, one is singular and the other is nonsingular.

As there is no involution in $F \setminus F'$, the subgroup $\bar{Z}$ lies in the Tits group $F'$. Now the Schur multiplier of $F'$ is trivial [6, p. 74], and thus the preimage of $F'$ in $H$ is the direct product of $T$ with a simple subgroup $F_0$ of $H$ isomorphic to $F'$. Correspondingly, there is a fours subgroup $Z$ of $F_0$ with $(T \times Z)/T = \bar{Z}$. We will show that $Z$ is a singular $2^2$-subgroup with $t \in Q_Z$. As $P$ induces $S_3$ on $\bar{Z}$, all involutions of $Z$ are conjugate.

As we can verify $(x_{1,2}x_{5,10})^8 = x_{21,24}$, every $2A$-involution of $F_0$ is a 8th power of some element of $M$. On the other hand, we show that there is no element $x$ of $M$ such that $x^8$ is a nonsingular involution of $M$. Suppose contrary and assume that $x^8 = t$ for some $x \in M$. Then the image $\bar{x}$ of $x$ in $B = H/T$ is an element of order 8 contained in the centralizer of a $2C$-involution $\bar{x}^4$ in $B$. As the centralizer has shape $(2^2 \times F_4(2))2$, the involution $\bar{x}^4$ of $B$ lies in $F_4(2)$. However, this implies that $x^4$ is an involution in $M$, by [14, Lemma 3.6(c)], which is a contradiction.

Thus we conclude that $2A$-involutions in $F_0 \cong {}^2F_4(2)'$ are singular involutions in $H$. In particular, for every involution $z$ of $Z$, $z$ is singular and $tz$ is nonsingular. Thus $t \in \bigcap_{z\in Z^\#} Q_z$.

We show that $Z$ is singular. As a subgroup of $P$ induces $S_3$ on $Z$, it suffices to show that $Z \leqslant Q_w$ for the singular involution $w$ of $F_0$ whose image in $B$ is the central involution $x_{21,24}$ in a Sylow 2-subgroup $U \cap F'$. It is easily verified that $\bar{Z} = \langle x_{21,24}, x_{17,23}\rangle$ is a second center of $U \cap F'$. Let $U_0$ is a subgroup of $F_0$ such that $(T \times U_0)/T = U \cap F'$. Then $U_0 \leqslant C_H(w)$. Take an element $g$ of order 3 lying in the preimage of the $S_4$-direct factor of $L_3$. As $g$ commutes with $F_0$ modulo $T$, we have $[g, F_0] = 1$ in $H$, in particular, $g$ lies in $C_H(w)$. Then its image in $C_H(w)/(H \cap Q_w)$ centralizes a 2-subgroup $U_0(H \cap Q_w)/(H \cap Q_w) \cong U_0/(U_0 \cap Q_w)$. Now there are two classes of elements of order 3 in $Co_2 \cong C_H(w)/(H \cap Q_w)$, with centralizers $3_+^{1+4}.2_-^{1+4}.A_5$ and $3 \times U_4(2).2$. As those centralizers have Sylow 2-subgroups of order $2^7$, we have $|U_0/(U_0 \cap Q_w)| \leqslant 2^7$, and hence $|U_0 \cap Q_w| \geqslant |U_0|/2^7 = 2^4$. Thus $(U_0 \cap Q_w)/\langle w\rangle$ is a nontrivial normal subgroup of $U_0/\langle w\rangle$ ($\leqslant C_H(w)/\langle w\rangle$). In particular, it intersects with $Z(U_0/\langle w\rangle)$ nontrivially. On the other hand, $Z(U_0/\langle w\rangle) = Z/\langle w\rangle \cong 2$, as $Z$ is the second center of $U_0$. Hence $Z/\langle w\rangle \leqslant (U_0 \cap Q_w)/\langle w\rangle$, that is, $Z \leqslant Q_w$, as desired.

(2) As there are odd number of Sylow 2-subgroups of $P$, there is a Sylow 2-subgroup $U$ of $P$ normalized by $g$. Then $U$ is a Borel subgroup of $L_2^\infty$ and we may assume that $g$ is the graph automorphism $\gamma$ by replacing $g$ by an element in $Ug$. We set $F := C_{L_2^\infty}(\gamma) \cong {}^2F_4(2)$. As $P$ is normalized by $\gamma$, $P = U$, $P_{14}$ or $P_{23}$. In the former two cases, $P\langle\gamma\rangle$ centralizes the central involution $x_{21,24}$ of $Z(U_{14}) \cap F$, which is a $2B$-involution in $B$ by Lemma 2. In the last case, $Z(U_{23}) \cap F = \langle x_{21,24}, x_{17,23}\rangle$, which is the image in $B$ of a singular $2^2$-subgroup $Z$ of $H$ with $t \in Q_Z$ by claim (1). Thus claim (2) follows. $\square$

### 1.3. Classification

Let $R$ be a radical 2-subgroup of $B$. By the list of maximal 2-local subgroups of $B$ obtained by [15] and [14], $N_B(R)$ is contained in $L_i$ for some $i = 1, \ldots, 8$ up to conjugacy.

We first study the radical subgroups $R$ such that $N_B(R)$ is contained in one of $L_1$, $L_2$ and $L_3$, but is not contained in $L_j$ for any $j \geqslant 4$.

Assume first that $N_B(R) \leqslant L_3$ but $R \neq V_3$. Let $S$ and $F$ be the direct factors of $L_3$ isomorphic to $S_4$ and ${}^2F_4(2)$, respectively. Then $R = R_S \times R_F$, where $R_S$ and $R_F$ are radical 2-subgroups of $S$ and $F$, respectively, allowing one of them to be trivial [22, Lemma 2]. If $R_F$ is nontrivial, then it is a unipotent radical of $F$, and $N_F(R_F)$ is a parabolic subgroup of $F$. By Lemma 3(1), the center of the unipotent radical $R_F$ is the image in $B$ of a singular $2^i$-subgroup $E_i$ of $H$ with $t \in Q_{E_i}$ for $i = 1$ or 2. As $N_B(R)$ ($\leqslant S \times F = L_3$) normalizes $R_F$ and thus its center, it normalizes $E_i$. As the image of $E_i$ is conjugate to $Z(V_{3+i})$, then $N_B(R)$ is contained in $L_{3+i}$ for $i = 1$ or 2 up to conjugacy.

Hence we may assume $R_F = 1$. Then the only possibility for $R = R_S$ is a Sylow 2-subgroup $D_8$ of the $S_4$-factor $S$, as $R$ contains $V_3$ properly. Conversely, if $R = V_3^{(d)}$ is such a subgroup, its preimage in $H$ is a semidihedral group $S_{16}$, which contains a unique $Q_8$-subgroup, the preimage of $V_3$ (see the description of $L_3$ in the last subsection). Hence $N_B(V_3^{(d)})$ normalizes $V_3$, and is contained in $L_3 = N_B(V_3)$. This implies that $V_3^{(d)}$ is in fact a radical 2-subgroup of $B$ [22, Lemma 1(2)].

Assume next that $N_B(R) \leqslant L_2$ but $R \neq V_2$. Then $R/V_2$ is a radical 2-subgroup of $L_2/V_2 \cong F_4(2).2$. We may also assume that $N_B(R)$ is not contained in $L_1$. Then $N_B(R)$ contains an element outside $L_1 \cap L_2 = V_2 \times L_2^\infty$, where $L_2^\infty = C_B(V_2) \cong F_4(2)$. Let $U := R \cap L_2^\infty$. Suppose $U \neq 1$. Then $N_{L_2^\infty}(U)$ is a parabolic subgroup of $L_2^\infty$. As $N_B(R)$ contains an element $g$ of $L_2$ outside $V_2 \times L_2^\infty$, the parabolic subgroup $V_2 N_{L_2^\infty}(U)/V_2$ of $(L_2/V_2)' \cong F_4(2)$ is normalized by the element of $L_2/V_2 \cong F_4(2).2$ outside $F_4(2)$, corresponding to $g$. Then we may apply Lemma 3(2) to conclude that $V_2 N_{L_2^\infty}(U)\langle g\rangle/V_2$ normalizes a singular $2^i$-subgroup $E_i$ in $H$ with $t \in Q_{E_i}$ for $i = 1$ or 2. As $N_B(R)/V_2$ is a subgroup of $V_2 N_{L_2^\infty}(U)\langle g\rangle/V_2$, it also normalizes $E_i$. The image of $E_i$ is conjugate to $Z(V_{3+i})$, and thus $N_B(R) \leqslant L_{3+i}$ for $i = 1$ or 2 up to conjugacy.

Thus we may assume that $U = 1$. Then $(R/V_2)^\#$ is an involution of $L_2/V_2 \cong F_4(2).2$ outside $(L_2/V_2)' \cong F_4(2)$. As there is a single class of such involutions [5, (19.5)], $R/V_2$

is conjugate to $V_3^{(d)}/V_2$ (see the description of $L_2$ and $L_3$ in the last paragraph). Thus we do not have any new radical subgroup in this case.

Now we assume that $N_B(R) \leqslant L_1$ but $R \neq V_1$. Then $R/V_1$ is a radical 2-subgroup of $L_1/V_1 \cong {}^2E_6(2).2$. If $R/V_1$ intersects $(L_1/V_1)' \cong {}^2E_6(2)$ nontrivially, then the intersection is a unipotent radical of $(L_1/V_1)'$. We now apply [14, Lemma 3.2] to the parabolic subgroups of $(L_1/V_1)'$. Then $N_B(R)$ normalizes the singular subgroup $Z(V_4) \cong 2$, $Z(V_5) \cong 2^2$, $Z(V_6) \cong 2^3$ or the subsection $Z(V_8) \cong 2^9$ of an ark up to conjugacy. Hence $N_B(R) \leqslant L_j$ for some $j = 4, 5, 6$ or 8 in this case. If $R/V_1$ is generated by an outer involution of ${}^2E_6(2)$, it has the centralizer $F_4(2)$ in $(L_1/V_1)'$, as the other two classes have the centralizer with nontrivial $O_2$-part and hence $R/V_1$ is not radical [22, Lemma 3]. However, as we saw in the description of $L_2$, this implies that $R$ is conjugate to $V_2$. Thus we do not obtain any new radical subgroups.

In the remaining cases, we have $N_B(R) \leqslant L_j$ for some $j = 4, \ldots, 8$ up to conjugacy. Now $L_j$ is the normalizer of $Z(V_j)$, which corresponds to a singular subgroup $E$ with $t \in Q_E$ or an ark containing $t$. Hence, up to here, we have verified the following, which is used in the classification of the radical 2-subgroups of the Monster.

**Lemma 4.** *For a radical* 2*-subgroup $R$ of $B$, one of the following holds*:

(1) *$N_B(R)$ normalizes either a singular subgroup $E$ of $H$ with $t \in Q_E$ of order $2^i$ for some $i = 1, 2, 3$ or an ark containing $t$.* (*In this case $N_B(R) \leqslant L_j$ for $j = 4, 5, 6$ or* 8 *up to conjugacy.*)
(2) *$R$ is conjugate to one of $V_i$ $(i = 1, 2, 3)$ or $V_3^{(d)}$.*

We will now examine the cases where $N_B(R)$ is contained in some maximal 2-local subgroup of characteristic-2 type.

Assume first that $N_B(R) \leqslant L_7$. The center $Z := Z(V_7)$ is a singular $2^5$-subgroup lying in the class $\mathcal{S}(5, 1)$ with $t \in Q_Z$. Suppose $R \neq V_7$. Then $V_7 < R$ and $R/V_7$ is a radical 2-subgroup of $L_7/V_7 \cong L_5(2)$, acting naturally on $Z(V_7)$. Thus it is a unipotent radical of $L_5(2)$. According to the initial term of the flag indexing the unipotent radical $R/V_7$, the centralizer in $Z(V_7)$ of $R/V_7$ is a singular $2^i$-subgroup for $i = 1, 2, 3$ or 4. As the centralizer is normalized by $N_B(R)$, we have $N_B(R) \leqslant L_{3+i}$ for $i = 1, 2, 3$ in the former three cases. In the last case, $R/V_7 \cong 2^4$ is the unipotent radical corresponding to a hyperplane, and the preimage of the centralizer $C_{Z(V_7)}(R/V_7)$ in $H$ is the direct product of $T$ and a singular $2^4$-subspace $E$ of $Z(V_7)$. By Lemma 1, there is a unique ark $V$ containing $T$ and $E$. Hence, $N_B(R)$ normalizes $V$ and $N_B(R) \leqslant L_8$ up to conjugacy.

Next assume that $N_B(R) \leqslant L_8$ but $R \neq V_8$. Then $V_8 < R$ and $R/V_8$ is a radical 2-subgroup of $L_8/V_8 \cong Sp_8(2)$. Let $V$ be the ark with $V/T = Z(V_8)$. Then $L_8/V_8$ acts naturally on $t^{\perp}/T \cong 2^8$, where $t^{\perp}$ is the hyperplane of $V$ perpendicular to $t$ with respect to the symmetric form associated with the quadratic form on $V$ preserved by $N_M(V)/C_M(V) \cong \Omega_{10}^{+}(2)$ (see the description of $L_8$ in the last subsection). Then $U/V_8$ is a unipotent radical of $Sp_8(2)$, indexed by a flag of totally isotropic subspaces of $t^{\perp}/T$. Thus the centralizer $C_{t^{\perp}/T}(R/V_8)$ is an $i$-dimensional totally isotropic subspace for $i = 1, 2, 3, 4$, according to the initial term of the flag. As a totally isotropic $i$-subspace of $t^{\perp}/T$ corresponds to a singular $2^i$-subgroup of $t^{\perp}$, if the centralizer is $i$-dimensional for $i = 1, 2, 3$, then $N_B(R)$

normalizes a totally singular $2^i$-subspace, and hence $N_B(R) \leqslant L_{3+i}$. If $i = 4$, then $R/V_8 \cong 2^{10}$ corresponds to a maximal totally isotropic subspace and $N_{L_8/V_8}(R/V_8) \cong L_4(2)$. Then $C_{t^\perp/T}(R/V_8)$ is the image in $B$ of a singular $2^4$-subspace $E$ of $V$ with $t \in E$. Applying Lemma 1 again, we conclude that $N_B(R) \leqslant L_8$, the normalizer of $V$. Thus from [22, Lemma 1(2)] we obtain one new radical subgroup $V_8^{(4)}$ with $V_8^{(4)} \sim V_8.2^{10}$ and the automizer $N_B(V_8^{(4)})/V_8^{(4)} \cong L_4(2)$ in this case.

We assume that $N_B(R) \leqslant L_6$ but $R \neq V_6$. Then $R/V_6$ is a radical 2-subgroup of $L_6/V_6 \cong S_5 \times L_3(2)$, in which $S_5$ and $L_3(2)$ acts, respectively, trivially and naturally on $Z(V_6) \cong 2^3$. Then it follows from [21, Lemma 2] that $R/V_6 = (R_t/V_6) \times (R_n/V_6)$ for some radical 2-subgroups $R_t/V_6$ of $S_5$ and $R_n/V_6$ of $L_3(2)$, allowing that one of them is trivial. If $R_n/V_6 \neq 1$, the centralizer $C_{Z(V_6)}(R_n/V_6)$ is a $2B$-pure $2^i$-subgroup for some $i = 1, 2$, which corresponds to a singular $2^i$-subgroup. Thus the centralizer is conjugate to $Z(V_{3+i})$, and $N_B(R) \leqslant L_{3+i}$ up to conjugacy. Thus we may assume that $R_n = 1$. Then $R_t/V_6$ is conjugate to one of the three representatives of radical 2-subgroups of $S_5$ [22, Lemma 4.1(1)]. Conversely, for any such group $R_t/V_6$, the center of $R_t$ coincides with $Z(V_6) \cong 2^3$, as the $S_5$-direct factor corresponds to $C_B(Z(V_6))/V_6$. Then $N_B(R_t) \leqslant N_B(Z(R_6)) = L_6$ and $R_t$ is in fact a radical subgroup of $B$ [22, Lemma 1(2)]. Thus we have three new classes of radical subgroups with center conjugate to $Z(V_6) \cong 2^3$.

By the same argument as above, we obtain seven new classes of radical subgroups with center conjugate to $Z(V_5) \cong 2^2$, with representatives $R_m$ such that $R_m/V_6$ are representatives of radical 2-subgroups of $C_{L_5}(Z(V_5))/V_5 \cong M_{22}.2$ [21, Lemma 11].

Table 1
Radical 2-subgroups of the Baby Monster

| $R$ | $R \sim$ | $Z(R)$ | $N(R)/R$ | $R$ | $R \sim$ | $Z(R)$ | $N(R)/R$ |
|---|---|---|---|---|---|---|---|
| $V_7$ | $2^5[2^{25}]$ | $2B^{31}$ | $L_5(2)$ | $V_4$ | $2_+^{1+22}$ | $2B^1$ | $Co_2$ |
| $V_8$ | $2^9 2^{16}$ | $2^9$ | $S_8(2)$ | $V_4^{(C)}$ | $V_4 2^{10}$ | 2 | $M_{22}2$ |
| $V_8^{(4)}$ | $V_8 2^{10}$ | $2B^{15}$ | $L_4(2)$ | $V_4^{(Q)}$ | $V_4 2^{4+10}$ | 2 | $S_3 \times S_5$ |
| $V_6$ | $2^3[2^{32}]$ | $2B^7$ | $S_5 \times L_3(2)$ | $V_4^{(Q*)}$ | $V_4[2^{15}]$ | 2 | $S_5$ |
| $V_6^{(1)}$ | $V_6 2$ | $2^3$ | $S_3 \times L_3(2)$ | $V_4^{(O')}$ | $V_4(2^{1+6} \times 2^4)$ | 2 | $L_4(2)$ |
| $V_6^{(2)}$ | $V_6 2^2$ | $2^3$ | $S_3 \times L_3(2)$ | $V_4^{(O',l)}$ | $V_4^{(O')} 2^4$ | 2 | $S_3 \times S_3$ |
| $V_6^{(3)}$ | $V_6 D_8$ | $2^3$ | $L_3(2)$ | $V_4^{(O',l\pi)}$ | $V_4^{(O')} 2^4 2$ | 2 | $S_3$ |
| $V_5$ | $2^2 2^{10} 2^{20}$ | $2B^3$ | $M_{22}2 \times S_3$ | $V_4^{(O',\pi)}$ | $V_4^{(O')} 2^3$ | 2 | $L_3(2)$ |
| $V_5^{(Q)}$ | $V_5 2^3 2$ | $2^2$ | $S_5 \times S_3$ | $V_4^{(H)}$ | $V_4 2^{1+8}$ | 2 | $S_6(2)$ |
| $V_5^{(O')}$ | $V_5 2^4 2$ | $2^2$ | $L_3(2) \times S_3$ | $V_4^{(H,p)}$ | $V_4^{(H)} 2^5$ | 2 | $S_4(2)$ |
| $V_5^{(QO')}$ | $V_5 2^4 2^2$ | $2^2$ | $S_3 \times S_3$ | $V_4^{(H,l)}$ | $V_4^{(H)} 2^{1+6}$ | 2 | $S_3 \times S_3$ |
| $V_5^{(H)}$ | $V_5 2^4$ | $2^2$ | $S_6 \times S_3$ | $V_4^{(H,\pi)}$ | $V_4^{(H)} 2^6$ | 2 | $L_3(2)$ |
| $V_5^{(HQ)}$ | $V_5^{(H)} 2^2 2$ | $2^2$ | $S_3 \times S_3$ | $V_4^{(H,pl)}$ | $V_4^{(H)} 2^5 2^2$ | 2 | $S_3$ |
| $V_5^{(HO')}$ | $V_5^{(H)} 2^2 2$ | $2^2$ | $S_3 \times S_3$ | $V_4^{(H,p\pi)}$ | $V_4^{(H)} 2^5 2^2$ | 2 | $S_3$ |
| $V_5^{(HQO')}$ | $V_5^{(H)} 2^3 2$ | $2^2$ | $S_3$ | $V_4^{(H,l\pi)}$ | $V_4^{(H)} 2^6 2^2$ | 2 | $S_3$ |
| | | | | $V_4^{(H,pl\pi)}$ | $V_4^{(H)}[2^9]$ | 2 | 1 |
| $V_3$ | $2^2$ | $= 2C^3$ | $S_3 \times {}^2F_4(2)$ | $V_1$ | 2 | $= 2A^1$ | ${}^2E_6(2)2$ |
| $V_3^{(d)}$ | $D_8$ | $2C^1$ | ${}^2F_4(2)$ | $V_2$ | $2^2$ | $= 2A^2C^1$ | $F_4(2)2$ |

Finally, there is a bijection between the classes of radical 2-subgroups $R$ of $B$ with $N_B(R) \leqslant L_4$ and those of $L_4/V_4 \cong Co_2$, as $V_4 \cong 2_+^{1+22}$ is extraspecial [22, Lemma 1(4)]. Hence we have 15 new radical subgroups with center conjugate to $Z(V_4) \cong 2$ in this last case, by [21, Theorem 19].

Summarizing, we have in total 4 classes of radical 2-subgroups whose normalizers are not of characteristic-2 type (Lemma 4), while we have $1, 1+1=2, 1+3=4, 1+7=8$ and $1+15=16$ classes in the case where $N_B(L_i) \leqslant L_i$ for $i = 7, 8, 6, 5$ and 4, respectively. Thus in total we have $4+(1+2+4+8+16) = 4+31 = 35$ classes of radical 2-subgroups of the Baby Monster $B$. The shapes of representatives and its automizers modulo $V_i$ for $i = 5, 4$ are already described in [21], from which Table 1 is easy to obtain.

**Theorem 5.** *There are exactly* 35 *classes of radical* 2*-subgroups of the Baby Monster, among which* 31 (*except those represented by* $V_1$, $V_2$, $V_3$ *and* $V_3^{(d)}$) *are centric. Description of the structures of the representatives and their normalizers is given in Table* 1.

## 2. Radical 2-subgroups of the Monster

### 2.1. Brief description of maximal 2-locals

There are just two classes of involutions in the Monster, called $2A$ and $2B$, with centralizers of shapes $2 \cdot B$ and $2_+^{1+24} \cdot Co_1$ respectively, where $B$ denotes the Baby Monster. It follows from [15, Theorem 1] and [14, Theorem A] that every 2-local subgroup of $M$ is contained in one of the following seven subgroups up to conjugacy:

$L_1 \sim 2 \cdot B$;
$L_2 \sim 2^2 \cdot {}^2E_6(2).S_3$;
$L_3 \sim 2_+^{1+24} \cdot Co_1$;
$L_4 \sim 2^2.2^{11}.2^{22}.(M_{24} \times S_3)$;
$L_5 \sim 2^3.2^6.2^{12}.2^{18}.(3 \cdot S_6 \times L_3(2))$;
$L_6 \sim 2^5.2^{10}.2^{20}.(S_3 \times L_5(2))$; and
$L_7 \sim 2^{10+16} \cdot \Omega_{10}^+(2)$.

We set $V_i := O_2(L_i)$ ($i = 1, \ldots, 7$). By maximality of $L_i$, each $V_i$ is a radical 2-subgroup of $M$. Moreover, $L_i = N_M(Z(V_i))$ for all $i = 1, \ldots, 7$.

These maximal local subgroups are described as follows. We freely use the notation given in the first subsection for the Baby Monster, in particular, singular subgroups and arks. There are exactly 6 classes of singular subgroups. They are of order up to $2^5$, and for each $i = 1, \ldots, 4$, singular $2^i$-subgroups form a single conjugacy class. There are two classes of singular $2^5$-subgroups in $M$, denoted $\mathcal{S}(5,1)$ and $\mathcal{S}(5,2)$. The center $V := Z(V_7) \cong 2^{10}$ is an ark and $L_7$ is its normalizer in $M$. Let $(Z_1, Z_2, Z_3, Z_4)$ be a flag of totally singular $i$-subspaces $Z_i$ of $V$ ($i = 1, \ldots, 4$) with respect to the quadratic form on $V$ preserved by $L_7/C_M(V) \cong \Omega_{10}^+(2)$. Then $Z_i$ represents a class of singular $2^i$-subgroups of $M$ ($i = 1, \ldots, 4$). We may take $Z_i$ to be the center $Z(V_{2+i})$ of $V_{2+i}$ for $i = 1, 2, 3$.

Let $Z_5^{(1)}$ and $Z_5^{(2)}$ be the two singular 5-subspaces of $V$ containing $Z_4$. Then $Z_5^{(k)}$ represents the class $\mathcal{S}(5,k)$ of singular $2^5$-subgroups of $M$ for both $k=1$ and 2. We may take $Z_5^{(2)} = Z(V_6)$. As a $2^4$-singular subgroup is contained in a unique subgroup in $\mathcal{S}(5,2)$ [15, Corollary 4.12], the normalizer $N_M(Z_4)$ is contained in $L_6 = N_M(Z_5^{(2)})$. As a member of $\mathcal{S}(5,1)$ is contained in a unique ark [15, Lemma 5.10], the normalizer $N_M(Z_5^{(1)})$ is contained in $L_7 = N_M(V)$.

As for the non-characteristic-2 type maximal locals $L_1$ and $L_2$, they are, respectively, the normalizers of $2A$-pure $2^j$-subgroups for $j = 1, 2$. Their structures are examined to some extent in the last section.

### 2.2. Classification

Now we start classification of the radical 2-subgroups of the Monster. It is very quick to complete, since those related to non-characteristic-2 type maximal locals are already treated in the last section.

Let $R$ be a radical 2-subgroup of $M$, the Monster. It follows from the classification of maximal 2-local subgroups of $M$ by [15, Theorem 1] and [14, Theorem B] that $N_M(R) \leqslant L_i$ for some $i = 1, \ldots, 7$ up to conjugacy.

We first study the radical subgroups $R$ such that $N_M(R)$ is contained in one of $L_1$ and $L_2$, but is not contained in $L_j$ for any $j \geqslant 3$.

Assume first that $N_M(R) \leqslant L_2$ but $R \neq V_2$. The group $V_2$ is a $2A$-pure fours subgroup and corresponds to the group denoted $\tilde{V}_1$ in the last section. By [22, Lemma 1(1)], $R$ properly contains $V_2$ and $R/V_2$ is a radical 2-subgroup of $L_2/V_2 \cong {}^2E_6(2)S_3$. Suppose $R/V_2$ intersects with $(L_2/V_2)^\infty \cong {}^2E_6(2)$ nontrivially, and let $U/V_2$ be the intersection. Then $U/V_2$ is a radical 2-subgroup of ${}^2E_6(2)$, and hence it is the unipotent radical of a parabolic subgroup. Thus the center $Z(U/V_2)$ is a subgroup of order $2^i$ for some $i = 1, 2, 3$ or 8, depending on the initial term of the flag of the ${}^2E_6(2)$-building indexing the parabolic subgroup. In [14, Lemma 3.2(d)], it is shown that if $i = 1, 2, 3$ then $Z(U/V_2)$ is $E_iV_2/V_2$ for a unique singular $2^i$-subgroup $E_i$ (perpendicular to $V_2$ in an ark containing them), and if $i = 8$, $Z(U/V_2) = V/V_2$ for an ark $V$ containing $V_2$. Thus in these cases, as $N_M(R)$ normalizes $U/V_2$ and $Z(U/V_2)$, we conclude that $N_M(R)$ is contained in $N_M(E_i)$ for $i = 1, 2, 3$ or $N_M(V)$. As the latter normalizers are conjugate to $L_{2+i}$ ($i = 1, 2, 3$) and $L_7$, we are reduced to the other cases. Thus we may assume that $R/V_2$ intersects with $(L_2/V_2)^\infty \cong {}^2E_6(2)$ trivially. As $R/V_2$ is radical, then it corresponds to an involution in ${}^2E_6(2)S_3$ outside ${}^2E_6(2)$ such that its centralizer in ${}^2E_6(2)$ has the trivial $O_2$-part [22, Lemma 3]. There are exactly two classes of outer involutions for ${}^2E_6(2)$ and only one of them (with centralizer $F_4(2)$) satisfies this condition [5, (19.9)(iii)]. Thus at most one class of new radical subgroups is obtained in this case. Conversely, let $R$ be the inverse image in $L_2$ of a subgroup of order 2 inducing on $(L_2/V_2)^\infty \cong {}^2E_6(2)$ an automorphism with centralizer $F_4(2)$. Then $R$ corresponds to the inverse image $\tilde{V}_2 \cong D_8$ of the radical group '$V_2$' in the last section. As its centralizer in $M$ is the centralizer in $C_M(V_2) \sim V_2{}^2E_6(2)$ of an outer involution, we have $N_M(R) \sim V_2(F_4(2) \times 2)$ and $O_2(N_M(R)) = R$. Thus $R \cong D_8$ (denoted $V_2^{(d)}$) is in fact a radical subgroup of $M$.

Next assume that $N_M(R) \leqslant L_1$ but $R \neq V_1$ ($T$ with the notation in the last section). Then $R/T$ is a radical 2-subgroup of the Baby Monster $L_1/T$. It follows from Lemma 4 that either $N_M(R)$ is contained in $L_{2+i}$ for $i = 1, 2, 3$ or $L_7$, or $R/V_1$ is conjugate to one of the four explicit subgroups. In the latter cases, $R$ is the inverse image $\tilde{V}_i$ of the subgroup '$V_i$' in $B$ ($i = 1, 2, 3$) or $V_3^{(d)}$. In view of the descriptions of non-characteristic-2 type maximal locals of the Baby Monster, given in the last section, with the notation there, it is immediate to see that $\tilde{V}_1 = V_2$, $\tilde{V}_2 = V_2^{(d)}$ (obtained in the last step), $\tilde{V}_3 = A \cong Q_8$ and that the inverse image of $V_3^{(d)}$ is the semidihedral group $SD_{16}$. Conversely, as the centralizers of these subgroups are contained in $C_M(T) = H$, their normalizers are the preimage in $H$ of their normalizers in $H/T$ for the last three cases. Then they are in fact radical subgroups. Hence, we obtain two new classes of radical 2-subgroups other than $V_1$ in this case, represented by $A = V_1^{(q)} \cong Q_8$ and $V_1^{(sd)} \cong SD_{16}$.

Summarizing, we have in total $(1+1) + (1+2) = 5$ classes of radical 2-subgroups (all of them are noncentric), which correspond to those for the Baby Monster plus the class of $V_1 = T$.

We now shift to the maximal 2-locals of characteristic-2 type.

First consider the case where $N_M(R) \leqslant L_7$ but $R \neq V_7$. Then $R/V_7$ is a unipotent radical of a parabolic subgroup of $\Omega_{10}^+(2) \cong L_7/V_7$ acting naturally on the ark $Z(V_7) \cong 2^{10}$. Hence the centralizer $C_{Z(V_7)}(R/V_7)$ is a singular $i$-subspace for some $i = 1, \dots, 5$, according to the initial term of the flag of the $\Omega_{10}^+(2)$-building indexing the parabolic subgroup $N_M(R/V_7)$. Assume $i = 1, 2$ or $3$. As a singular $i$-subspace of an ark $Z(V_7)$ is conjugate to $Z(V_{2+i})$, $N_M(R)$ normalizes $Z(V_{i+2})$ up to conjugacy, and we are reduced to the case $N_M(R) \leqslant N_M(Z(V_{i+2})) = L_{i+2}$. Assume $i = 4$. Then the flag consists of the two $2^5$-singular subspaces containing the singular 4-subspace $C_{Z(V_7)}(R/V_7)$, in which one, say $X$, belongs to the class $\mathcal{S}(5, 2)$. As every singular $2^4$-subgroup is contained in a unique member of $\mathcal{S}(5, 2)$ [15, Corollary 4.12], $N_M(R)$ normalizes the unique member $X$ of $\mathcal{S}(5, 2)$ containing $C_{Z(V_7)}(R/V_7)$. Thus up to conjugacy we are reduced to the case $N_M(R) \leqslant N_M(X) = L_6$. If $i = 5$ and $C_{Z(V_7)}(R/V_7)$ belongs to the class $\mathcal{S}(5, 2)$, the same conclusion holds. Thus the unique remaining possibility is that $R/V_7$ is the unipotent radical corresponding to a singular 5-subspace of the first class. Conversely, if $R$ is such a 2-subgroup, then $C_{Z(V_7)}(R/V_7) = Z(R)$ is a member of $\mathcal{S}(5, 1)$, which is contained in a unique ark [15, Lemma 5.10]. Thus $N_M(R)$ normalizes the unique ark $Z(V_7)$ containing $Z(R)$, and then $N_M(R) \leqslant N_M(Z(V_7)) = L_7$. By [22, Lemma 1(2)], this implies that $R$ is in fact a radical 2-subgroup of $M$. Thus, in this case, we have one new class of radical 2-subgroups, represented by a subgroup $V_7^{(5.1)}$ of shape $V_7.2^{10}$ with center $2^5$ (in the class $\mathcal{S}(5, 1)$) and normalizer $V_7.2^{10}.L_5(2)$.

Next we treat the case $N_M(R) \leqslant L_{2+i}$ for $i = 4, 3$ or $2$, with some uniformity. In this case, $L_{2+i}/V_{2+i}$ is the direct product of a group $D_t$ centralizing $Z(V_{2+i})$ and a group $D_n \cong GL(Z(V_{2+i}))$ acting naturally on $Z(V_{2+i})$. Remark that $Z(V_{2+i})$ is a singular subgroup isomorphic to $2^5$, $2^3$ or $2^2$, according to $i = 4, 3$ or $2$. Then we have $R/V_{2+i} = R_t \times R_n$, where $R_t$ and $R_n$ are radical 2-subgroups of $D_t$ and $D_n$ respectively, allowing one of them to be trivial [22, Lemma 3]. Remark first that if $R_n = 1$, then every $R$ for which $R/V_{2+i} = R_t$ is a radical 2-subgroup of $D_t$ is in fact a radical 2-subgroup of $M$. This follows from [22, Lemma 1(2)] with the following observation: as $N_M(R)$ normalizes $Z(R) = C_{Z(V_{2+i})}(R/V_{2+i}) = Z(V_{2+i})$, we have $N_M(R) \leqslant L_{2+i} = N_M(Z(V_{2+i}))$. Thus we have new classes of radical subgroups of this type, corresponding to the classes of radical 2-subgroups of $D_t$ [22, Lemma 1(3)]. Remark that all those radical subgroups

$R$ have the center $Z(V_{2+i})$ and the normalizer of shape $V_{2+i}.(N_{D_t}(R_t) \times D_n)$. According to $i = 4, 3$ or 2, we have $D_t = S_3$, $3S_6$ or $M_{24}$, which has 1, 6 and 13 classes of radical 2-subgroups respectively [21, Lemma 7]. The unique representative for $S_3$ is 2. The representatives for $M_{24}$ are indexed by the following symbols ($O$, $T$, $S$ and $*$ correspond respectively to an octad, trio, sextet and a truncated node in the 2-local geometry for $M_{24}$), among which the six symbols properly containing $S$ correspond to the classes of radical 2-subgroups of $3S_6$:

$$O, \quad T, \quad S, \quad O*, \quad T*, \quad OT, \quad OS, \quad TS,$$
$$OTS, \quad OT*, \quad OS*, \quad TS*, \quad OTS*.$$

Using these symbols, the representatives are denoted by $V_6^{(1,0)}$ for $i = 4$, $V_5^{(X,0)}$ ($X = O$, $T$, $OT$, $O*$, $T*$, $OT*$) for $i = 3$, and $V_4^{(X,0)}$ with $X$ in the above 13 symbols for $i = 2$. Thus we have in total $1 + 6 + 13 = 20$ new classes of radical 2-subgroups with $R_n = 1$.

On the other hand, if $R_n \neq 1$, the centralizer $C_{Z(V_{2+i})}(R) = C_{Z(V_{2+i})}(R_n)$ is a proper singular subgroup of $Z(V_{2+i})$.

We now separate the cases: we first consider the case $i = 4$. Then $R_n$ is a unipotent radical of $L_5(2)$ and the centralizer $C_{Z(V_6)}(R) = C_{Z(V_6)}(R_n)$ is a $j$-subspace for some $j = 1, \ldots, 4$. If $j = 1, 2, 3$, then the singular $j$-subspace $C_{Z(V_6)}(R)$ is conjugate to $Z(V_{2+j})$. Thus $N_M(R)$ normalizes $Z(V_{2+j})$ and $N_M(R) \leqslant L_{2+j}$ up to conjugacy. In the remaining case where $j = 4$, $R_n \cong 2^4$ and $C_{Z(V_6)}(R)$ is a singular $2^4$-subgroup. As $Z(V_6)$ is the unique member of $\mathcal{S}(5, 2)$ containing $C_{Z(V_6)}(R)$ [15, Corollary 4.12], we have $N_M(R) \leqslant L_6 = N_M(Z(V_6))$. This implies that such an $R$ is in fact a radical subgroup of $M$ [22, Lemma 1(2)]. As there are exactly two classes of radical 2-subgroups of $S_3$, including the trivial subgroup, we then have two new classes of radical 2-subgroups, represented by subgroups $V_6^{(0,4)}$ and $V_6^{(1,4)}$ of shapes $V_6(1 \times 2^4)$ and $V_6(2 \times 2^4)$ with the same center $2^4$ and normalizers $V_6(1 \times 2^4)(S_3 \times L_4(2))$ and $V_6(2 \times 2^4)(1 \times L_4(2))$, respectively.

Next consider the case $i = 3$. Then $R_n$ is a unipotent radical of $L_3(2)$ and the centralizer $C_{Z(V_5)}(R) = C_{Z(V_5)}(R_n)$ is a $j$-subspace for $j = 1$ or 2. Thus $N_M(R)$ is contained in $L_{2+j} = N_M(C_{Z(V_5)}(R))$ up to conjugacy, and we are reduced to the case $N_M(R) \leqslant L_3$ or $L_4$. In the case $i = 2$, $C_{Z(V_4)}(R)$ is a singular group of order 2, and we are reduced to the case where $N_M(R) \leqslant L_3$. Thus no new radical subgroup is obtained.

Finally, we are reduced to the case $N_M(R) \leqslant L_3$, where $V_3 \cong 2_+^{1+24}$ is extraspecial. Hence it follows from [22, Lemma 1(4)] that the classes of radical 2-subgroups $R$ with $N_M(R) \leqslant L_3$ distinct from $V_3$ bijectively correspond to the classes of radical 2-subgroups of $L_3/V_3 \cong Co_1$, which are determined by Sawabe [17].

Summarizing, we have $1 + 1 = 2$ classes in $L_7$ (represented by $V_7$ and $V_7^{(5.1)}$), $1 + 1 + 2 = 4$ classes in $L_6$ (represented by $V_6$, $V_6^{(1,0)}$, $V_6^{(0,4)}$ and $V_6^{(1,4)}$), $1 + 6 = 7$ classes in $L_5$ (represented by $V_5$ and $V_5^X$ for the six symbols $X$ above) and $1 + 13 = 14$ classes in $L_4$ (represented by $V_4$ and $V_4^X$ for the 13 symbols above). Furthermore, as there are 30 classes of radical 2-subgroups of $Co_1$ [17, Main result], we have $1 + 30 = 31$ classes in $L_2$, adding $V_2$ itself. Thus we find in total $2 + 4 + 7 + 14 + 31 = 58$ classes of radical 2-subgroups in maximal 2-locals of characteristic-2 type, which are all centric in view of their centralizers.

Table 2
Radical 2-subgroups of the Monster

| $R$ | $R\sim$ | $Z(R)$ | $N(R)/R$ | $R$ | $R\sim$ | $Z(R)$ | $N(R)/R$ |
|---|---|---|---|---|---|---|---|
| $V_7$ | $2^{10+16}$ | $2^{10}=A^{496}B^{527}$ | $\Omega_{10}^+(2)$ | $V_3$ | $2_+^{1+24}$ | $2=B^1$ | $Co_1$ |
| $V_7^{(5.1)}$ | $V_7 2^{10}$ | $2^5=B^{31}$ | $L_5(2)$ | $V_3^E$ | $V_1 2^{11}$ | $2$ | $M_{24}$ |
| $V_6$ | $2^5 2^{10} 2^{20}$ | $2^5=B^{31}$ | $S_3\times L_5(2)$ | $V_3^Q$ | $V_3 2^{4+12}$ | $2$ | $S_3\times 3S_6$ |
| $V_6^{(1,0)}$ | $V_6 2$ | $2^5$ | $L_5(2)$ | $V_3^{QS}$ | $V_3^Q 2$ | $2$ | $3S_6$ |
| $V_6^{(0,4)}$ | $V_6 2^4$ | $2^4=B^{15}$ | $S_3\times L_4(2)$ | $V_3^{Q_1}$ | $V_3 2^{2+12}$ | $2$ | $S_3\times L_4(2)$ |
| $V_6^{(1,4)}$ | $V_6(2\times 2^4)$ | $2^4$ | $L_4(2)$ | $V_3^{Q_1^{(1)}}$ | $V_3^{Q_1} 2^3$ | $2$ | $S_3\times L_3(2)$ |
| $V_5$ | $2^3 2^6 2^{12} 2^{18}$ | $2^3=B^7$ | $3S_6\times L_3(2)$ | $V_3^{Q_1^{(2)}}$ | $V_3^{Q_1} 2^4$ | $2$ | $S_3^3$ |
| $V_5^{(O)}$ | $V_5 2^2$ | $2^3$ | $S_3^2\times L_3(2)$ | $V_3^{Q_1^{(3)}}$ | $V_3^{Q_1} 2^3$ | $2$ | $S_3\times L_3(2)$ |
| $V_5^{(T)}$ | $V_5 2^2$ | $2^3$ | $S_3^2\times L_3(2)$ | $V_3^{Q_1^{(12)}}$ | $V_3^{Q_1}[2^5]$ | $2$ | $S_3\times S_3$ |
| $V_5^{(OT)}$ | $V_5[2^3]$ | $2^3$ | $S_3\times L_3(2)$ | $V_3^{Q_1^{(23)}}$ | $V_3^{Q_1}[2^5]$ | $2$ | $S_3\times S_3$ |
| $V_5^{(O*)}$ | $V_5[2^3]$ | $2^3$ | $S_3\times L_3(2)$ | $V_3^{Q_1^{(13)}}$ | $V_3^{Q_1} 2_+^{1+4}$ | $2$ | $S_3$ |
| $V_5^{(T*)}$ | $V_5[2^3]$ | $2^3$ | $S_3\times L_3(2)$ | $V_3^{Q_1^{(123)}}$ | $V_3^{Q_1}[2^6]$ | $2$ | $1$ |
| $V_5^{(OT*)}$ | $V_5[2^4]$ | $2^3$ | $L_3(2)$ | $V_3^R$ | $V_3 2^{1+8}$ | $2$ | $O_8^+(2)$ |
| $V_4$ | $2^2 2^{11} 2^{22}$ | $2^2=B^3$ | $M_{24}\times S_3$ | $V_3^{R^{(1)}}$ | $V_3^R 2^6$ | $2$ | $L_4(2)$ |
| $V_4^{(O)}$ | $V_4 2^4$ | $2^2$ | $L_4(2)\times S_3$ | $V_3^{R^{(2)}}$ | $V_3^R 2^6$ | $2$ | $L_4(2)$ |
| $V_4^{(T)}$ | $V_4 2^6$ | $2^2$ | $S_3 L_3(2)\times S_3$ | $V_3^{R^{(3)}}$ | $V_3^R 2^6$ | $2$ | $L_4(2)$ |
| $V_4^{(S)}$ | $V_4 2^6$ | $2^2$ | $3S_6\times S_3$ | $V_3^{R^{(4)}}$ | $V_3^R 2_+^{1+8}$ | $2$ | $S_3^2\times S_3$ |
| $V_4^{(OT)}$ | $V_4 2^{3+4}$ | $2^2$ | $L_3(2)\times S_3$ | $V_3^{R^{(12)}}$ | $V_3^R 2^3$ | $2$ | $L_3(2)$ |
| $V_4^{(OS)}$ | $V_4 2^{2+6}$ | $2^2$ | $S_3^2\times S_3$ | $V_3^{R^{(13)}}$ | $V_3^R 2^3$ | $2$ | $L_3(2)$ |
| $V_4^{(O*)}$ | $V_4 2_+^{1+6}$ | $2^2$ | $L_3(2)\times S_3$ | $V_3^{R^{(14)}}$ | $V_3^R 2^4$ | $2$ | $S_3\times S_3$ |
| $V_4^{(TS)}$ | $V_4 2^{4+4}$ | $2^2$ | $S_3^2\times S_3$ | $V_3^{R^{(23)}}$ | $V_3^R 2^3$ | $2$ | $L_3(2)$ |
| $V_4^{(T*)}$ | $V_4[2^8]$ | $2^2$ | $S_3^2\times S_3$ | $V_3^{R^{(24)}}$ | $V_3^R 2^4$ | $2$ | $S_3\times S_3$ |
| $V_4^{(OTS)}$ | $V_4[2^9]$ | $2^2$ | $S_3\times S_3$ | $V_3^{R^{(34)}}$ | $V_3^R 2^4$ | $2$ | $S_3\times S_3$ |
| $V_4^{(OT*)}$ | $V_4[2^9]$ | $2^2$ | $S_3\times S_3$ | $V_3^{R^{(123)}}$ | $V_3^R 2_+^{1+4}$ | $2$ | $S_3$ |
| $V_4^{(OS*)}$ | $V_4[2^9]$ | $2^2$ | $S_3\times S_3$ | $V_3^{R^{(124)}}$ | $V_3^R[2^5]$ | $2$ | $S_3$ |
| $V_4^{(TS*)}$ | $V_4[2^9]$ | $2^2$ | $S_3\times S_3$ | $V_3^{R^{(134)}}$ | $V_3^R[2^5]$ | $2$ | $S_3$ |
| $V_4^{(OTS*)}$ | $V_4[2^{10}]$ | $2^2$ | $S_3$ | $V_3^{R^{(234)}}$ | $V_3^R[2^5]$ | $2$ | $S_3$ |
| | | | | $V_3^{R^{(1234)}}$ | $V_3^R[2^6]$ | $2$ | $1$ |
| | | | | $V_3 V$ | $V_3 2^2$ | $2$ | $(3\times G_2(4))2$ |
| | | | | $V_3 V2$ | $V_3 V2$ | $2$ | $G_2(4)2$ |
| | | | | $V_3 F$ | $V_3 2^2$ | $2$ | $(S_3\times U_3(3))2$ |
| $V_2$ | $2^2$ | $2^2=A^3$ | ${}^2E_6(2)S_3$ | $V_1$ | $2$ | $2=A^1$ | $B$ |
| $V_2^{(d)}$ | $D_8$ | $2=A^1$ | $F_4(2)2$ | $V_1^{(q)}$ | $Q_8$ | $2$ | $S_3\times {}^2F_4(2)$ |
| | | | | $V_1^{(sd)}$ | $SD_{16}$ | $2$ | ${}^2F_4(2)$ |

Together with 5 classes of noncentric radicals, we have now completed the enumeration of $5 + 58 = 63$ classes of radical 2-subgroups of the Monster.

**Theorem 6.** *There are exactly* 63 *classes of radical* 2*-subgroups of the Monster, among which* 58 (*except those represented by* $V_1$, $V_1^{(q)}$, $V_1^{(sd)}$, $V_2$ *and* $V_2^{(d)}$) *are centric. Description of the structures of the representatives and their normalizers is given in Table* 2.

## Appendix

In this section, a list of references (Table 3) to earlier work on the radical 2-subgroups of sporadic simple groups is given, although the list may be incomplete. Even though it is easy to determine the radical 2-subgroups for $J_2$ and $McL$, we are unaware of any published reference on those groups. Thus the lists (Tables 4, 5) are given here for completeness. (According to Uno, Dade verified his remarkable conjecture to $J_2$ and $Ru$ in the final and the ordinary form respectively, but Dade's notes seem to be unpublished. As for $McL$, Murray's thesis [16] is unpublished, and Entz-Pahlings [9] examine elementary abelian

Table 3
Radical 2-subgroups of the sporadic simple groups

| | |
|---|---|
| $M_{11}$ | An and Conder [1] |
| $M_{12}$ | An and Conder [1] |
| $M_{22}$ | An and Conder [1], Yoshiara [21, Lemma 13] |
| $M_{23}$ | An and Conder [1] |
| $M_{24}$ | An and Conder [1], Yoshiara [21, Lemma 7] |
| $J_2$ | Dade (see Table 4) |
| $Suz$ | Yoshiara [23, Proposition 2] |
| $HS$ | Hassen and Horvath [10], Yoshiara [22, Proposition 12] |
| $McL$ | Murray [16] (see Table 5) |
| $Co_3$ | An [3] |
| $Co_2$ | Yoshiara [21, Theorem 19] |
| $Co_1$ | Sawabe [17] |
| $F_{22}$ | An and O'Brien [4], Kitazume and Yoshiara [13, Theorem 15] |
| $F_{23}$ | An and O'Brien [4] |
| $F_{24}'$ | Kitazume and Yoshiara [13, Theorem 12] |
| $He$ | An [2] |
| $HN$ | Yoshiara [22, Theorem 18] |
| $Th$ | Yoshiara [21, Theorem 20] |
| $BM$ | Theorem 5 in this paper |
| $M$ | Theorem 6 in this paper |
| $J_1$ | Dade [8, (10.5)] |
| $J_3$ | Kotlica [12] |
| $Ru$ | Sawabe [18] |
| $O'N$ | Yoshiara [22, Theorem 15] |
| $Ly$ | Yoshiara [22, Proposition 7] |
| $J_4$ | Yoshiara [21, Theorem 17] |

Table 4
Radical 2-subgroups of $J_2$

| $R$ | $R\sim$ | $Z(R)$ | $N(R)/R$ | $R$ | $R\sim$ | $Z(R)$ | $N(R)/R$ |
|---|---|---|---|---|---|---|---|
| $F_2$ | $2^2$ | $=B^3$ | $3\times A_5$ | $E_2$ | $2^{2+4}$ | $2^2=A^3$ | $3\times S_3$ |
| $F_2^{(2)}$ | $2^4$ | $=A^3B^{12}$ | $3^2$ | $E$ | $2_-^{1+4}$ | $2=A^1$ | $A_5$ |
| | | | | $E^{(1)}$ | $E2^2$ | 2 | 3 |

Table 5
Radical 2-subgroups of *McL*

| $R$ | $R\sim$ | $Z(R)$ | $N(R)/R$ | $R$ | $R\sim$ | $Z(R)$ | $N(R)/R$ |
|---|---|---|---|---|---|---|---|
| $E_1$ | $2^4$ | $=A^{15}$ | $A_7$ | $Z_l$ | $2_+^{1+4}$ | $2=A^1$ | $S_3\times S_3$ |
| $E_2$ | $2^4$ | $=A^{15}$ | $A_7$ | $Z_{pl}$ | $2[2^5]$ | 2 | $S_3$ |
| $E_1^{(2)}$ | $E_12^2$ | $2^2=A^3$ | $S_3\times S_3$ | $Z_{l\pi}$ | $2[2^5]$ | 2 | $S_3$ |
| $Z$ | 2 | $=A^1$ | $A_8$ | $Z_{pl\pi}$ | $2[2^6]$ | 2 | 1 |

2-subgroups only. I was informed that Dade's conjecture is verified by O'Brien and others for $Fi_{22}$, $Fi_{23}$, *HN*, $J_4$, $O'N$, *Ru* and some Conway groups. The classification of radical 2-subgroups is carried out with the aid of GAP. For the precise information, see O'Brien's home page: http://www.math.auckland.ac.nz/.)

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